

Dynamics of Birational Plane Mappings. The Arnold complexity difference equation.

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Abstract

We consider a dynamics of a generic birational plane map $\Phi_n : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$, \mathbb{CP}^2 -image of the birational mapping (inverse map is also rational) $F_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ and its such important characteristic as the Arnold complexity $C_A(k)$, which is proportional $d(k) = \deg(\Phi_n^k)$ - a degree of k -iteration of the map Φ_n , on the basis on algebraic-geometrical properties of such maps. Additional importance of this characteristic follows from the Veselov conjecture about the polynomial boundedness of the growth of $d(k)$ for integrable dynamical systems with a discrete time defined by birational plane maps. The autonomous linear difference equation with integer coefficients for $d(k)$ is obtained. This equation is fully defined by σ_1 nonnegative integers m_1, \dots, m_{σ_1} that are determined by relations: $\Phi_n^{-m_i}(O_{\alpha_i}) = O_{\beta_i}^{(-1)}$, $i \in (1, 2, \dots, \sigma_1)$, where $\Phi_n^{-m_i}$ is m_i -iteration of inverse map, $O_{\alpha_i}, O_{\beta_i}^{(-1)}$, $\alpha_i, \beta_i \in (1, 2, \dots, \sigma)$ are indeterminacy points of the direct and inverse maps, $\sigma_1 \leq \sigma$ and σ is a number of indeterminacy points of Φ_n, Φ_n^{-1} . If σ_1 is equal to zero that $d(k) = n^k$, otherwise the growth of $d(k)$ is fully defined by a root spectrum of the secular equation associated with the difference equation for $d(k)$. The Veselov conjecture corresponds to the root spectrum consisting of values being equal to modulo one. The author doesn't suppose that the reader has acquaintance with the algebraic geometry (AG) in \mathbb{CP}^2 and the dynamical systems theory (DST) or the functional equations since in the paper there are given all needed definitions of used concepts of AG or DST and theorems.

1 Introduction. Set of the problem and Main Result.

Let consider the system of birational functional equations (BFEs) for functions $y(w) : \mathbf{C} \rightarrow \mathbf{C}^N$ in one complex variable w of the form

$$y(w+1) = F_n(y(w)), \quad y(w) : \mathbf{C} \rightarrow \mathbf{C}^N, \quad w \in \mathbf{C}, \quad F_n \in \mathbf{Bir}(\mathbf{C}^N). \quad (1)$$

For $w = m \in \mathbf{Z}$ the above BFEs are a dynamical system with discrete time or cascade. Here the map $F_n : y \mapsto y' = F_n(y) = \frac{f_i(y)}{f_{N+1}(y)}, i = (1, 2, \dots, N)$, $f_i(y)$ for $\forall i$ are polynomials in y , $\deg F_n(y) = \max_{i=1}^{N+1} \{\deg(f_i(y))\} = n$, is a given birational map of the group of all automorphisms of $\mathbf{C}^N \rightarrow \mathbf{C}^N$ (the Cremona group or $\mathbf{Bir}(\mathbf{C}^N)$) with the coefficients from \mathbf{C} .

A preliminary investigation of the dynamics of the mapping F_n is important in the context of consideration of the integrability problem of dynamical systems or the BFEs of the form (1). Such consideration is more convenient and effective to realize in \mathbf{CP}^N . Let us give the definition of the mapping Φ_n , the

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image of the map F_n in \mathbf{CP}^N , at $N = 2$ since below we shall mainly consider the dynamics of plane birational mappings. At the transition to \mathbf{CP}^2 $y \mapsto z : y_i = z_i/z_3, i = (1, 2)$ the maps F_n, F_n^{-1} transform into the maps Φ_n, Φ_n^{-1} :

$$\Phi_n : \quad z \mapsto z', \quad z'_1 : z'_2 : z'_3 = \phi_1(z) : \phi_2(z) : \phi_3(z), \quad z, z' \in \mathbf{CP}^2, \quad (2)$$

$$\phi_i(z) = z_3^n f_i(z_l/z_3), \quad i \in (1, 2, 3), \quad l = (1, 2), \quad (3)$$

and $\phi_i(z)$ are homogeneous polynomials in z without any common factors. The map

$$\Phi_n^{-1} : z' \mapsto z, \quad z_1 : z_2 : z_3 = \phi_1^{(-1)}(z') : \phi_2^{(-1)}(z') : \phi_3^{(-1)}(z'), \quad z, z' \in \mathbf{CP}^2 \quad (4)$$

is defined analogously.

In the abstract and above we used such familiar concepts as "integrability", "integrable maps", "integrable dynamical systems", and "integrable functional equations". In order to avoid different understanding of these terms, we shall below give our definition of these concepts. The integrability problem for dynamical systems and functional equations is solved if we obtain for the BFEs (1) the family of first integrals $I(y(w), w)$ of dimension of $1 \leq m \leq N - 1$ of the form:

$$I(y(w), w) = c(w), c(w) : \mathbf{C} \rightarrow \mathbf{C}^m, \quad c(w+1) = c(w), \quad (5)$$

$$I(y, w) : (\mathbf{C}^N \otimes \mathbf{C}) \rightarrow \mathbf{C}^m, \quad (6)$$

where arbitrary periodic functions $c(w)$ in w parameterize the level lines of first integrals.

In addition to these m first integrals we always have one more first transitive integral parameterized by the periodic function

$$w \rightarrow w + \beta(w), \quad \beta(w) : \mathbf{C} \rightarrow \mathbf{C}, \quad \beta(w+1) = \beta(w). \quad (7)$$

If integer $m = N - 1$, we can speak about full integrability, otherwise partial integrability of equation (1) at $1 \leq m < N - 1$. We have a general solution of equations BFEs (1) if we obtain the solution of BFEs (1) in the explicit form $y = Y(w, c(w))$ with $c(w) : \mathbf{C} \rightarrow \mathbf{C}^{N-1}$. If $I(y, w)$ is a rational function of (y, w) or a rational function of y and fraction-linear one in variables $\tau(w)$ where $\tau(w)$ are variables of the form λ^w , then we can speak about the algebraic integrability of BFEs (1). If $I(y, w) \in \text{Hol}(y, w)$, i.e. $I(y, w)$ is a holomorphic function of variables, then we can speak about non-algebraic integrability of BFEs (1). If the algebraic integrability is the subject for using algebra-geometrical methods, then non-algebraic integrability is the subject for using classical methods and theorems from the theory of dynamical systems due to H. Poincare, C.L. Siegel, G.D. Birkhoff, A.N. Kolmogorov, V.I. Arnold, J. Moser, D.V. Anosov and others (see Arnold and Il'yashenko (1988), Anosov et al. (1988)), and also using classical results of the number theory and the transcendental number theory (see, for example, (Moser, 1994), (Rerikh, 1995a), (Rerikh, 1997), (Rerikh, 1998b) as examples of using classical results of A. Baker (Baker, 1990), (Baker, 1966), (Baker, 1967a,b, 1968), (Baker, 1971), (Baker and Wüstholtz, 1993) and N.I. Feldman (Feldman, 1982), (Feldman, 1968)). In the paper, we shall not discuss the integrability problem in more detail. There are also other definitions of the concept integrability. The A.P. Veselov definition as applied to the dynamical systems of the form (1) acting in the plane (bipolynomial maps, (Veselov, 1989), (Veselov, 1991)) is as follows: "The map F_n is integrable if there exist another map Ψ for which $\Psi^m \neq F_n^k \forall k, m \in \mathbf{Z}$, where Ψ^m is the m th iteration of the map Ψ but F_n^k is the k th iteration of the map F_n , commuting with F_n : $\Psi \circ F_n \equiv F_n \circ \Psi$ ". The Moser definition means the existence of a holomorphic map $H(y) : u = H(y), \quad \mathbf{C}^N \rightarrow \mathbf{C}^N$ that transforms the maps F_n to its linear part $H \circ F_n \circ H^{-1} \equiv A \stackrel{\text{def}}{=} \frac{\partial F_n}{\partial y}|_{y=y_0}$, where y_0 is a fixed point of the map F_n but the matrix A defines the linear part of the map F_n . Such a definition is natural for the theory of dynamical systems. (see examples of non-algebraically integrable dynamical systems (Rerikh, 1992), (Rerikh, 1995b) (Rerikh, 1995a), (Rerikh, 1997), (Rerikh, 1998a)). This Moser definition of integrability is in fact a local concept in the neighbourhood of a fixed point of a map as well as a concept of a local non-integrability in a neighbourhood of a fixed point of a map. (See

the Moser example (Moser, 1960) of non-integrable cubic bipolynomial map in the neighbourhood of the zero elliptic fixed point.)

V.I. Arnold in papers (Arnold, 1990b), (Arnold, 1990a) introduced and investigated such a characteristic of a dynamical system as the topological *complexity* of the intersection of a submanifold X of manifold M , moved by a dynamical system, a smooth mapping $A : M \rightarrow M$ with the other given compact smooth submanifold Y of M :

$$Z_k = (A^k X) \cup Y.$$

In the simplest case for plane mappings Φ the complexity $C_A^\Phi(k) \equiv Z_k$ can be defined (Veselov, 1992) as the number of intersection points of a fixed curve Γ_1 with the image of another curve Γ_2 under the k^{th} iteration of Φ :

$$C_{A;\Gamma_1\Gamma_2}^\Phi(k) = \#(\Gamma_1 \cap \Phi^k(\Gamma_2)).$$

If the mapping Φ is a birational one from the group **BirCP**² and the curves Γ_1, Γ_2 are algebraic curves in **CP**², then it is easy to see that the growth of $C_{A;\Gamma_1\Gamma_2}^\Phi(k)$ will in general be as follows:

$$C_{A;\Gamma_1\Gamma_2}^\Phi(k) = \deg(\Gamma_1)\deg(\Gamma_2)d_\Phi(k) \leq \deg(\Gamma_1)\deg(\Gamma_2)(\deg\Phi)^k,$$

where $d_\Phi(k) = \deg(\Phi^k)$ is the degree of the mapping $\Phi^k = \Phi \circ \Phi \circ \dots \circ \Phi$, which agrees well with general Arnold's results for smooth mappings and diffeomorphisms (Arnold, 1990b), (Arnold, 1990a).

The Arnold complexity was found to be an important characteristic in the context of the integrability of such dynamical systems. A.P. Veselov introduced in (Veselov, 1991), (Veselov, 1992) the conjecture about a polynomial growth of the Arnold complexity $d(k)$ with k for integrable plane birational mappings and proved it for integrable bipolynomial ones in (Veselov, 1989) ($d(k)$ is bounded by a constant). To be more exact we reformulate the A.P. Veselov conjecture as " all integrable birational mappings have a polynomially bounded growth of the Arnold complexity $d(k)$ on k ".

The validity of the A.P. Veselov conjecture was also confirmed for many concrete integrable mappings by different researchers so that it is actual to prove it in a general case. This paper is the first step in this direction. The aim of this paper is to discuss the dynamics of generic birational plane mappings in the frames of their algebraic-geometrical properties and obtain the autonomous linear difference equation for the Arnold complexity $d(k)$.

The main result of the paper is the obtained autonomous linear difference equation for $d(k)$. This equation is fully defined by σ_1 nonnegative integers m_1, \dots, m_{σ_1} that are determined by relations: $\Phi_n^{-m_i}(O_{\alpha_i}) = O_{\beta_i}^{(-1)}$, $i \in (1, 2, \dots, \sigma_1)$, where $\Phi_n^{-m_i}$ is the m_i -iteration of the inverse map, $O_{\alpha_i}, O_{\beta_i}^{(-1)}$, $\alpha_i, \beta_i \in (1, 2, \dots, \sigma)$, are indeterminacy points of the direct and inverse maps, $\sigma_1 \leq \sigma$ and σ is a number of indeterminacy points of Φ_n, Φ_n^{-1} . If σ_1 is equal to zero, then $d(k) = n^k$, otherwise the growth of $d(k)$ is fully defined by a root spectrum of the secular equation associated with the difference equation for $d(k)$. The A.P. Veselov conjecture corresponds to the root spectrum consisting of values being equal to modulo one. Thus, this equation gives the possibility to present all sets of numbers m_1, \dots, m_{σ_1} corresponding to integrable mappings if the A.P. Veselov conjecture is true.

In the following section, we shall perform a brief excursus into the theory of the Cremona transformations in the plane following (Hudson, 1927), Snyder et al. (1970), (Iskovskikh and Reid, 1991), (Shafarevich, 1977), (Coble, 1961). In Section 3 we introduce a new notion –the decomposition of sets of the indeterminacy points of direct Φ_n and inverse Φ_n^{-1} maps. Then in Section 4 we obtain the main equations of the dynamics of a generic birational map and the difference equation for the Arnold complexity $d(k)$. Different examples for illustration of Sections 2- 4 are set in Appendices A, B.

2 Brief excursus into the algebraic geometry

Let $z = (z_1, z_2, z_3)$ be a point of the projective plane **CP**². Let us consider a general curve of degree μ $f_\mu(z)$ defined by the equation

$$f_\mu(z) = \sum_{|l|=\mu} c_l z^l = 0, \quad l \stackrel{\text{def}}{=} (l_1, l_2, l_3), \quad |l| \stackrel{\text{def}}{=} l_1 + l_2 + l_3, \quad (8)$$

which has, in general, $\frac{\mu(\mu+3)}{2}$ free parameters.

2.1 Linear systems of curves

Definition 1 Let $P = (z_1^*, z_2^*, z_3^*)$ be a point of the curve (8) and let z_3^* be the coordinate of the point P which is nonzero, but, therefore, we can assign $z_3^* = 1$ as a result of the change $P \mapsto P/z_3^*$. The point P is called an r -fold one of the curve (8) if $f_\mu(z)$ has the following form in the system of coordinates z' : $z'_1 = z_1 - z_1^* z_3, z'_2 = z_2 - z_2^* z_3, z'_3 = z_3$

$$f_\mu(z) = f'_\mu(z') = \sum_{k=r}^{\mu} z'_3^{\mu-k} u_k(z'_1, z'_2), \quad (9)$$

$$r \stackrel{\text{def}}{=} \text{mult}(f_\mu(z))|_{z=z^*}, \quad \text{mult} \stackrel{\text{def}}{=} \text{multiplicity}, \quad (10)$$

where $u_k(z')$ are homogeneous polynomials of degree k in variables z'_1, z'_2 , but the first function $u_r(z')$ in expansion (9) defines r tangents for the curve at the point P .

An r -fold point imposes $\frac{r(r+1)}{2} = \sum_{i=0}^{r-1} (i+1)$ conditions ensured for a curve (8) of the form (9) and is called a simple, double, triple one, if r is equal 1, 2, 3 and so on. \diamond

Let us give a definition of a linear system of curves which is important in what follows.

Definition 2 (For more details see Snyder et al. (1970) and also all references therein on results and notions reviewed here.) The system of plane curves f_μ of degree μ is represented by an equation of the form

$$f_\mu = \sum_{i=1}^{k+1} c_i f_i(z) = 0, \quad (11)$$

where the functions $f_i(z)$ are homogeneous polynomials from $z = (z_1, z_2, z_3)$ of the same order μ and are linearly independent, is a linear system of curves (LSC) of dimension k .

Definition 3 A point $B_j^{(r_j)}$ which is at least an r_i -fold one for each curve of the system is called an r_i -fold basis point (see Definition 1) but a join of all basis points is called a basis set or a base of the LSAC $\mathbf{B} = \bigcup_j B_j^{(r_j)}$, where $j \in 1, 2, \dots, N_B$. Thus, the base is fully defined by two sets: the set of basis points and the set of theirs multiplicities on the LSAC which are linked with each other, as it is set above.

Counting separately the conditions imposed by all basis points necessary for reduction from the general curve (8) to the LSAC (11), we have the virtual dimension

$$K = \frac{\mu(\mu+3)}{2} - \sum_{j=1}^{N_B} \frac{r_j(r_j+1)}{2}.$$

In certain cases the conditions are not independent so that the effective dimension is

$$k = K + s, \quad (12)$$

where s is the number of independent relations among the linear conditions imposed by the base B on curves of order μ . A system for which $s = 0$ is said to be regular, otherwise irregular with irregularity (superabundance) s .

The effective genus p of a general curve of the irreducible system coincides with virtual P and is

$$p = P = \frac{(\mu-1)(\mu-2)}{2} - \sum_{j=1}^{N_B} \frac{r_j(r_j-1)}{2}. \quad (13)$$

For reducible curves the effective genus p equals

$$p = P + c - 1, \quad (14)$$

where c is a number of components of a reducible curve. The number of variable intersections of two curves of the system is the grade D

$$D = \mu^2 - \sum_{j=1}^{N_B} r_j^2. \quad (15)$$

The numbers K, D and P satisfy the relation

$$K = D - P + 1,$$

so that

$$k = D - p + s + 1$$

for an irreducible system, where $s \leq p$, because $D \geq (k - 1)$. The numbers D, p and k are invariant under birational mappings. A linear system of dimension $k = 1, 2, 3$ is called a pencil, a net and a web, respectively (see also Remark 1). \triangleleft

Remark 1 Definition 2 can be extended to the case of the linear systems of curves of which the system of the basis r -fold points includes some non-ordinary (extra ordinary) r -fold ones (see Definition 4 below).

Definition 4 The r -fold basis point of the linear system of curves is called a non-ordinary one if at this point the linear system of curves satisfies some additional tangency conditions as the existence of $r_1, 1 \leq r_1 \leq r$, common tangents (r -fold point of a simple contact) or the existence of some fixed curve touching upon these common tangents and osculating with each curve of the system (r -fold point of higher contact). Each r -fold non-ordinary point can be represented by the system of infinitely near ordinary points and be resolved using the technics of resolution of singularities of plane curves (see (Hudson, 1927), Chapter VII and also below Section 2.5).

Let us give a definition of a birational mapping $\Phi_n : \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^2$.

2.2 Definition of Birational Map, Noether theorem and Quadratic maps

Definition 5 *Birational map.* A mapping $\Phi_n : z \mapsto z'$, $z, z' \in \mathbb{C}\mathbb{P}^2$ in (2), where ϕ_i are homogeneous polynomials in $z, i = (1, 2, 3)$, of degree n , is called a birational mapping if it assigns one-to-one correspondence between z and z' , while the inverse mapping is given by (4) and it is also rational (genus $p = 0$), ϕ'_i being also homogeneous polynomials in z' , moreover, ϕ_i and $\phi_i^{(-1)}$ have no common factors.

Associated with Φ_n and Φ_n^{-1} the linear systems of curves $\phi, \phi^{(-1)}$ of dimension $k = 2$, genus $p = 0$ and grade $D = 1$

$$\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3, \quad (16)$$

$$\phi^{(-1)} = c_1^{(-1)}\phi_1^{(-1)} + c_2^{(-1)}\phi_2^{(-1)} + c_3^{(-1)}\phi_3^{(-1)} \quad (17)$$

(for $c_i, c_i^{(-1)} \in \mathbb{C}$) are fully given by theirs bases $\mathbf{B}, \mathbf{B}^{-1}$ (for bases of LSACs associated with maps we shall use symbols $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{O}, \mathbf{B}^{-1} \stackrel{\text{def}}{=} \mathbf{O}^{-1}, r_\alpha \stackrel{\text{def}}{=} i_\alpha, r_\beta^{(-1)} \stackrel{\text{def}}{=} i_\beta^{(-1)}, \alpha \in (1, 2, \dots, \sigma \stackrel{\text{def}}{=} N_{\mathbf{B}}), \beta \in (1, 2, \dots, \sigma^{(-1)} \stackrel{\text{def}}{=} N_{\mathbf{B}^{-1}}), (\sigma = \sigma^{(-1)})$.) define the first and second rational nets which are images of nets of lines. The basis points O_α, O_β^{-1} are indeterminacy ones for the maps Φ_n, Φ_n^{-1} and are called fundamental ones (or F-points). The equality of genus p to zero is a necessary and sufficient condition for the birationality of the rational map Φ_n (2). \triangleleft

Theorem 1 (*M. Noether*) Every Cremona plane mapping Φ_n (2) can be resolved into quadratic mappings

$$\Phi_n = C \circ Q_1 \circ Q_2 \cdots \circ Q_j,$$

where C is a collineation (linear mapping in \mathbb{CP}^2), but mappings Q_1, \dots, Q_j are quadratic ones. . \triangleleft

At the end, we should give the definition of the main object— the generators of the Cremona group, namely, birational quadratic mappings.

Definition 6 Any generic quadratic Cremona mapping is generated by a composition

$$\Phi_2 \equiv B^{-1} \circ I_s \circ B_1, \quad (18)$$

where

$$B : z \mapsto j^{(-1)} = Bz, \quad B_1 : z \mapsto j = B_1 z \quad (19)$$

are generic linear mappings from the $\text{PGL}(2, \mathbb{C})$ group and I_s is an involution, the standard Cremona mapping with three simple F -points $O_\alpha \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and three principal lines $J_\alpha = (z : j_\alpha(z) = 0) \in \{(z_1 = 0), (z_2 = 0), (z_3 = 0)\}$:

$$I_s : z \mapsto z' \quad z'_1 : z'_2 : z'_3 = z_2 z_3 : z_1 z_3 : z_1 z_2. \quad (20)$$

$$I_s : J_\beta \rightarrow O_\beta^{(-1)}, \quad O_\beta^{(-1)} \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \quad (21)$$

$$I_s^{(-1)} : z' \mapsto z \quad z_1 : z_2 : z_3 = z'_2 z'_3 : z'_1 z'_3 : z'_1 z'_2, \quad (22)$$

$$I_s^{(-1)} : J_\alpha^{(-1)} \rightarrow O_\alpha, \quad O_\alpha \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}. \quad (23)$$

In the triangular frame of reference (19) mapping (18) takes a very simple form

$$\begin{aligned} \Phi_2 : j(z) \mapsto j^{(-1)}(z') & \quad j_1^{(-1)}(z') : j_2^{(-1)}(z') : j_3^{(-1)}(z') = \\ & \quad j_2(z) j_3(z) : j_1(z) j_3(z) : j_1(z) j_2(z). \end{aligned} \quad (24)$$

The mapping Φ_2 (18) is specialized if two or three F -points are adjacent or infinitely near (Iskovskikh and Reid, 1991) and has, respectively, the following forms:

$$\Phi_{2a} \equiv B^{-1} \circ I_a \circ B_1, \quad I_a : z \mapsto z' \quad z'_1 : z'_2 : z'_3 = z_2^2 : z_1 z_2 : z_1 z_3, \quad (25)$$

$$\Phi_{2b} \equiv B^{-1} \circ I_b \circ B_1, \quad I_b : z \mapsto z' \quad z'_1 : z'_2 : z'_3 = z_1^2 : z_1 z_2 : (z_2^2 - z_1 z_3), \quad (26)$$

moreover, involutions I_a, I_b from (25), (26) can be resolved as a composition of two or four, but not fewer, general mappings (18), respectively (see (Hudson, 1927), chapter III, pp. 35,37). Any two members of the net (25) touch one another and have a fixed common tangent $j_1 \equiv z_1 = 0$, but ones of the net (26) have a fixed common tangent $j \equiv z_1$ and osculate a fixed conic $z_2^2 - z_1 z_3$. These tangency conditions are simulated by two or three infinitely near points, so as equations (31)-(35) remain correct.

2.3 Properties of Birational Mapping

Definition 7 Properties. The one-to-one correspondence for direct Φ_n and inverse Φ_n^{-1} mappings does not hold only at indeterminacy or fundamental points (F -points) $O_\alpha \in \mathbf{O}$, $O_\beta^{(-1)} \in \mathbf{O}^{(-1)}$, $\alpha, \beta = (1, 2, \dots, \sigma)$, i.e., common basis points of multiplicities $i_\alpha, i_\beta^{(-1)}$ for functions $\phi_k(z), \phi_k^{(-1)}(z)$, $k = (1, 2, 3)$, and the associated linear systems ϕ (16) and $\phi^{(-1)}$ (17), respectively, and on principal or exceptional curves or exceptional divisors $J_\beta, J_\alpha^{(-1)}$, $\alpha, \beta = (1, 2, \dots, \sigma)$,

$$J_\beta \stackrel{\text{def}}{=} \{z : j_\beta(z) = 0\}, \quad J_\alpha^{(-1)} \stackrel{\text{def}}{=} \{z : j_\alpha^{(-1)}(z) = 0\}, \quad \alpha, \beta = (1, \dots, \sigma), \quad (27)$$

where $j_\beta, j_\alpha^{(-1)}$ are homogeneous polynomials in z of degrees $i_\beta^{(-1)}, i_\alpha$, respectively, moreover, the points $O_\alpha, O_\beta^{(-1)}$ blow up into the curves $J_\alpha^{(-1)}, J_\beta$ of degrees $i_\alpha, i_\beta^{(-1)}$ and the curves $J_\alpha^{(-1)}, J_\beta$ blow down into the points $O_\alpha, O_\beta^{(-1)}$,

$$O_\alpha \leftrightarrow J_\alpha^{(-1)}, \quad \deg J_\alpha^{(-1)} = i_\alpha, \quad (28)$$

$$O_\beta^{(-1)} \leftrightarrow J_\beta, \quad \deg J_\beta = i_\beta^{(-1)}, \quad (29)$$

respectively (see the concept of σ -process of blowing up of singularities in the theory of ordinary differential equations (Arnold, 1988) and the Kodaira theorem in the algebraic geometry (Griffiths and Harris, 1978)). \triangleleft

Definition 8 A fundamental point is called ordinary if at this point there are no any additional tangency conditions. In special cases of non-ordinary (extra-ordinary) F-points (see Definition 4), tangency conditions of any two members of the associated linear systems are expressed as multiplicities of infinitely near points (Iskovskikh and Reid, 1991), or adjoint points in the terminology of (Hudson, 1927). Each r -fold non-ordinary point can be represented by the system of infinitely near ordinary points and be transformed into ordinary ones using the technics of resolution of singularities of plane curves (see (Hudson, 1927), Chapter VII, Theorem 3 below and examples : 1, 2 in Appendices A, B).

Theorem 2 Jacobian. The Jacobian J of the mapping Φ_n equals

$$J = \left\| \frac{\partial \phi_k}{\partial z_i} \right\| \sim \prod_{\alpha=1}^{\sigma} j_\alpha. \quad \deg J = 3n - 3 \quad (30)$$

The formula corresponds to a birational map with ordinary F -points but in the case of non-ordinary (infinitely near) points it remains correct if we assign to the infinitely near F -points the same factors j_α with multiplicities in accordance with a characteristic of the map (see below Definition 2 and also examples 1, 2 in Appendix A). The determination of the Jacobian is a very simple way to find the principal curves. The principal curves $J_\alpha(J_\beta^{(-1)})$ intersect each other only at fundamental points $O_\alpha(O_\beta^{(-1)})$.

Remark 2 Characteristic. The set of numbers

$$\text{char}(\Phi_n) = \{n; i_1, i_2, \dots, i_\sigma\}, \quad , i_1 \geq i_2 \geq \dots \geq i_\sigma,$$

where i_α are the multiplicities of all indeterminacy points O_α of the mapping Φ_n , including infinitely near ones, is called the characteristic of mapping Φ_n . We shall denote the infinitely near F -points by the star: i_α^* and O_α^* .

Next in simplicity after quadratic birational map with $\text{char}(\Phi_2) = \{2; 1, 1, 1\}$ is a cubic map with $\text{char}(\Phi_3) = \{3; 2, 1, 1, 1, 1\}$ and then two quartic maps with $\text{char}(\Phi_4) = \{4; 2, 2, 2, 1, 1, 1\}$ and $\{4; 3, 1, 1, 1, 1, 1, 1, 1\}$.

The general mapping with a given characteristic depends on $2\sigma + 8$ parameters.

\triangleleft

Remark 3 Characteristic numbers. Let $i_{\beta\alpha}^{(-1)}$ be the multiplicity of curve $J_\alpha^{(-1)}$ at point $O_\beta^{(-1)}$ and $i_{\alpha\beta}$ be that of curve J_β at O_α . Then we have the equality $i_{\alpha\beta} = i_{\beta\alpha}^{(-1)}$ and the following relations between numbers $i_\alpha, i_\beta^{(-1)}, i_{\alpha\beta}$, expressing certain geometrical facts (summing in the left column over α and in the right one over β from 1 to σ):

$$\sum i_\alpha = 3(n-1), \quad \sum i_\beta^{(-1)} = 3(n-1), \quad (31)$$

$$\sum i_\alpha^2 = n^2 - 1, \quad \sum i_\beta^{(-1)2} = n^2 - 1, \quad (32)$$

$$\sum i_{\alpha\beta} = 3i_\beta^{(-1)} - 1, \quad \sum i_{\alpha\beta} = 3i_\alpha - 1, \quad (33)$$

$$\sum i_\alpha i_{\alpha\beta} = i_\beta^{(-1)} n, \quad \sum i_\beta^{(-1)} i_{\alpha\beta} = i_\alpha n, \quad (34)$$

$$\sum i_{\alpha\beta} i_{\alpha\gamma} = i_\beta^{(-1)} i_\gamma^{(-1)} + \delta_{\beta\gamma}, \quad \sum i_{\alpha\beta} i_{\gamma\beta} = i_\alpha i_\gamma + \delta_{\alpha\gamma}. \quad (35)$$

The conditions (31), (32) mean that the associated linear systems (16), (17) have the grade $D = 1$, the genus $p = 0$, the dimension $k = 2$, and the superabundance $s = 0$. The conditions (33), (35) provide rationality of the curves $J_\beta, J_\alpha^{(-1)}$ (27), and that their degrees are $i_\beta^{(-1)}$ and i_α , respectively. In the case of non-ordinary F -points the total number of distinct F -points need not be the same for the direct (2) and inverse (4) mappings. In the special cases, if at some β in the left parts of equations (33) and (35) (at $\beta = \gamma$) j_β breaks up into $1 \leq \nu \leq i_\beta^{(-1)}$ components, then the left parts of these equations must be replaced by

$$\sum i_{\alpha\beta} = 3i_\beta^{(-1)} - \nu, \quad \sum i_{\alpha\beta}^2 = i_\beta^{(-1)^2} + \nu. \quad (36)$$

Analogous changes in the right parts of equations (33) and (35) at some α must be made.

The upper limit for σ of the total number of F -points is given by the following formula:

$$\sigma \leq 2n - 1, \quad \text{if } n > 1.$$

◇

2.4 Behaviour of algebraic curves and LSAC under the action of the birational map (2)

Remark 4 Consider properties of a general curve $f_\mu(z') = 0$ of degree μ under the mapping (2). By map (2), the curve $f_\mu(z')$ is mapped into the curve $f_\mu(\phi(z)) = f'_{\mu'}(z)$ of degree $\mu' = \mu n$; moreover, every point O_α which is i_α -fold on $\phi(z)$ is μi_α -fold on $f'_{\mu'}$. If $f_\mu(z')$ has multiplicities $\gamma_\beta^{(-1)}$ at points $O_\beta^{(-1)}$, then $(\deg(j_\beta) \equiv i_\beta^{(-1)})$

$$f_\mu(z') = f'_{\mu'}(z) \prod_{\beta=1}^{\sigma} j_\beta^{\gamma_\beta^{(-1)}}, \quad \mu' = \mu n - \sum_{\beta=1}^{\sigma} \gamma_\beta^{(-1)} i_\beta^{(-1)}; \quad (37)$$

moreover, $f'_{\mu'}$ has multiplicities γ'_α at O_α (see the meaning of $i_{\alpha\beta}$ in Remark 3):

$$\gamma'_\alpha = \mu i_\alpha - \sum_{\beta=1}^{\sigma} i_{\alpha\beta} \gamma_\beta^{(-1)}. \quad (38)$$

If $f_\mu(z) = 0$ is a general curve of a linear system of curves of dimension k but $f'_{\mu'}$ is its image under the map Φ_n (2), that genus p and dimension k of the LSAC are invariants.

2.5 Birational equivalence and resolution of singularities of plane curves

Let us introduce the definition of birationally equivalent mapping in \mathbb{CP}^2 .

Definition 9 A mapping $\Phi_n \in \text{Bir}\mathbb{CP}^2, z \rightarrow z' \sim \phi(z)$, is birationally equivalent (or conjugated in terminology of Hudson's book) to a mapping $U : y \rightarrow y', y' \sim u(y) y, y' \in \mathbb{CP}^2$, if there exists a birational mapping $V_m : z \rightarrow y, y \sim v(z)$ of degree m such that $\Phi_n \equiv V_m^{-1} \circ U \circ V_m$.

Due to the standard method of resolution of singularities of plane curves (see (Hudson, 1927), chapter VII, p.129) the following problems can be solved by applying a composition of the corresponding Cremona quadratic mappings:

1. to transform any non-ordinary multiple point into a net of simple points;
2. to resolve any non-ordinary multiple point into an equivalent set of ordinary multiple points;
3. to transform any algebraic curve into one having ordinary multiple points only;
4. to transform any linear system of algebraic curves into one having ordinary base points only.

As the consequence of the standard method of resolution of singularities of plane curves the following theorem is represented to be valid.

Theorem 3 Any mapping Φ_n (2) with non-ordinary F -points (see Definition 4) by the corresponding transformation of birational equivalence (see Definition 9) is transformed into some mapping $\Phi_{n'} = V^{-1} \circ \Phi_n \circ V$ with only ordinary F -points where degree of this mapping $n' \geq n$ but the mapping V is a composition of a necessary (for resolution of all infinitely near points) number of quadratic mappings. (see Definition 6) (see example in Appendix B) \triangleleft

For illustration of this Section we set the examples of quadratic and cubic maps with ordinary and non-ordinary indeterminacy points in Appendix A but the example of using Theorem 3 is in Appendix B.

Below we shall deal with mappings having only ordinary indeterminacy points supposing that maps with non-ordinary ones was previously replaced by birationally equivalent maps with the help of Theorem 3.

3 Decomposition of the set of indeterminacy points

Let us consider orbits of indeterminacy points $O_\alpha \in O$ and $O_\beta^{(-1)} \in O^{(-1)}$ relative to the action of the inverse Φ_n^{-1} (4) and the direct map Φ_n (2), respectively, and let us introduce the following definitions.

Definition 10 The orbit \mathcal{O}_z of a point z with respect to the mapping Φ_n^{-1} (4) is the set of points $\mathcal{O}_z^k = \Phi_n^{-k}(z) = (\Phi_n^{-1})^k(z)$, $k \in \mathbb{Z}^+$, where \mathbb{Z}^+ is the set of non-negative integers. The orbit $\mathcal{O}_z^{(-1)}$ of a point z with respect to Φ_n (2) is defined analogously, $\mathcal{O}_z^{(-1)k} = \Phi_n^k(z) = (\Phi_n)^k(z)$ and $\Phi_n^k(z) \stackrel{\text{def}}{=} \Phi_n(\Phi_n(\dots(z\dots)))$, $\Phi_n^{-k}(z) \stackrel{\text{def}}{=} \Phi_n^{-1}(\Phi_n^{-1}(\dots(z\dots)))$ (see, for example, (Arnold and Il'yashenko, 1988), (Arnold, 1988)).

Definition 11 If the number k of the points of the orbit \mathcal{O}_z of the point z with respect to the mapping Φ_n^{-1} (4) is finite, where non-negative integer k is a minimal integer defined by the condition

$$\mathcal{O}_z^k = \Phi_n^{-k}(z) = z, \quad k \in \mathbb{Z}^+,$$

then the periodic points $(\Phi_n^{-m}(z))$, $m = (0, 1, \dots, k-1)$ form the set $\mathcal{O}_z^{(\text{cycle})}$ – a cycle of index k or period k of the mapping $\Phi_n^{-1}(z)$, but z is a fixed point of the mapping $\Phi_n^{-k}(z)$. A cycle of index k of the mapping $\Phi_n(z)$ (2) is defined similarly with help of the changes: $\mathcal{O}_z \mapsto \mathcal{O}_z^{(-1)}$ and $\Phi_n^{-m}(z) \mapsto \Phi_n^m(z)$, (see (Arnold and Il'yashenko, 1988)).

Let us introduce the notion of a tail of the cycle.

Definition 12 Let us call a subset $\mathcal{O}_z^{(\text{tail})}$ of the set \mathcal{O}_z a tail of the length l of the cycle $\mathcal{O}_y^{(\text{cycle})}$ where non-negative integer l is a minimal integer defined by the condition

$$\mathcal{O}_z^{(\text{tail})} \text{ of length } l : \{y = \Phi_n^{-l}(z), \text{ where } y \in \mathcal{O}_y^{(\text{cycle})}\}, \quad l \in \mathbb{Z}^+,$$

so that the point z is the beginning of the tail, but the point y is the beginning of the cycle and does not belong to the tail.

Definition 13 Decomposition

Let Φ_n (2) be a mapping of characteristic $n; i_1, i_2, \dots, i_\sigma$ and Φ_n^{-1} (4) be the inverse mapping (see Definition-Theorem 5, Remark 2). Define (Rerikh, 1998a) the decomposition of the sets $O, O^{(-1)}$ of fundamental points $O_\alpha, O_\beta^{(-1)}$ of these mappings as follows:

$$O \equiv O^{(\text{rest})} \cup O^{(\text{int})}, \quad O^{(-1)} \equiv O^{(-1)(\text{rest})} \cup O^{(-1)(\text{int})}, \quad (39)$$

$$O^{(\text{rest})} \equiv O^{(\text{cycle})} \cup O^{(\text{tails})} \cup O^{(\text{inf})},$$

$$O^{(-1)(\text{rest})} \equiv O^{(-1)(\text{cycle})} \cup O^{(-1)(\text{tails})} \cup O^{(-1)(\text{inf})}. \quad (40)$$

Here: $\mathbf{O}^{(inf)}$ is a subset of fundamental points O_α with infinite orbits

$$\mathbf{O}^{(inf)} : [\Phi_n^{-k}(\mathbf{O}) \cap [\mathbf{O}^{(-1)} \cup \mathbf{O}]] = \emptyset \quad \text{at } \forall k \ 1 \leq k < \infty; \quad (41)$$

$\mathbf{O}^{(cycle)}$ is a subset of fundamental points O_α having cyclic orbits $\mathcal{O}_z^m, z \in \mathbf{O}$, of index m_α ; $\mathbf{O}^{(tails)}$ is a subset of fundamental points O_α belonging to the tails of the orbits of the subset $\mathbf{O}^{(cycle)}$, to the tails of the orbits of the subset $\mathbf{O}^{(int)}$ and to the tails of the orbits of the subset $\mathbf{O}^{(inf)}$.

The subsets $\mathbf{O}^{(-1)(int)}$ and $\mathbf{O}^{(int)}$ of the sets $\mathbf{O}^{(-1)}$ and \mathbf{O} are defined below.

Definition 14 The subsets $\mathbf{O}^{(-1)(int)}$ and $\mathbf{O}^{(int)}$ of the sets $\mathbf{O}^{(-1)}$ and \mathbf{O} are under construction in the following manner. Let $\Phi_n^{-k}(\mathbf{O}(k)), \Phi_n^k(\mathbf{O}^{(-1)}(k)), k \geq 0, \Phi_n^{-k}|_{k=0} \equiv \text{id}$ be k^{th} -iterations of punctual sets $\mathbf{O}(k), \mathbf{O}^{(-1)}(k), \mathbf{O}(0) \equiv \mathbf{O}, \mathbf{O}^{(-1)}(0) \equiv \mathbf{O}^{(-1)}$ under the action of inverse and direct maps Φ_n^{-1}, Φ_n . Let us introduce the punctual set

$$\begin{aligned} \mathbf{O}^{(int)}(k) &\stackrel{\text{def}}{=} \Phi_n^{-k}(\mathbf{O}(k)) \cap \mathbf{O}^{(-1)}(k), \\ \mathbf{O}^{(-1)(int)}(k) &\stackrel{\text{def}}{=} \Phi_n^k(\mathbf{O}^{(-1)}(k)) \cap \mathbf{O}(k), \quad k \geq 0 \end{aligned} \quad (42)$$

and define the construction of the set $\mathbf{O}(k)$ and $\mathbf{O}^{(-1)}(k)$ at $k \geq 1$ in the following manner:

$$\begin{aligned} \mathbf{O}(k) &\stackrel{\text{def}}{=} \mathbf{O}(k-1)/\mathbf{O}^{(int)}(k-1), \\ \mathbf{O}^{(-1)}(k) &\stackrel{\text{def}}{=} \mathbf{O}^{(-1)}(k-1)/\mathbf{O}^{(-1)(int)}(k-1), \quad k \geq 1. \end{aligned} \quad (43)$$

Then subsets $\mathbf{O}^{(int)}$ and $\mathbf{O}^{(-1)(int)}$ are defined with the help of (42) and (43) by

$$\mathbf{O}^{(int)} \stackrel{\text{def}}{=} \bigcup_{k=0}^{k=m} \mathbf{O}^{(int)}(k), \quad \mathbf{O}^{(-1)(int)} \stackrel{\text{def}}{=} \bigcup_{k=0}^{k=m} \mathbf{O}^{(-1)(int)}(k), \quad (44)$$

where positive integer m is defined by the condition

$$\mathbf{O}(m+1)/\mathbf{O}^{(rest)} \equiv \emptyset, \quad \mathbf{O}^{(-1)}(m+1)/\mathbf{O}^{(-1)(rest)} \equiv \emptyset,$$

where there subsets $\mathbf{O}^{(rest)}, \mathbf{O}^{(-1)(rest)}$ are defined above by (40). Let be

$$\#\mathbf{O}^{(int)} = \sigma_1. \quad (45)$$

The above construction establishes the one-to-one correspondence between σ_1 pairs of equivalent indeterminacy points of the subsets $\mathbf{O}^{(int)}$ and $\mathbf{O}^{(-1)(int)}$ constructed above as follows:

$$\Phi^{-m_j}(O_{\alpha_j}) \equiv O_{\beta_j}^{(-1)}, \quad j = (1, \dots, \sigma_1), \quad (46)$$

$$O_{\alpha_j} \in \mathbf{O}^{(int)}, \quad O_{\beta_j}^{(-1)} \in \mathbf{O}^{(-1)(int)}, \quad \alpha_j \text{ and } \beta_j \in (1, \dots, \sigma), \quad (47)$$

$$m_j \in (m_1, \dots, m_{\sigma_1}) \quad 0 \leq m_1 \leq \dots \leq m_{\sigma_1}, \quad (48)$$

where nonnegative integers m_j are lengths of the orbits of points O_{α_j} but integer m in equation (44) is equal to m_{σ_1} . \triangleleft

Remark 5 Since the coordinates of the indeterminacy points $O_{\alpha_j}, O_{\beta_j}^{(-1)}, j = (1, 2, \dots, \sigma_1)$ are functions of $2\sigma + 8$ parameters, equations (46)-(48) define in the space of $(2\sigma + 8)$ parameters $2\sigma_1$ subvarieties of dimension $2\sigma + 8 - 2\sigma_1$. If the A.P. Veselov conjecture is true, integrable mappings correspond to these subvarieties.

\triangleleft

4 Dynamics of a generic birational mapping. Difference equation for the Arnold complexity.

Theorem 4 defines the dynamics of birational mapping and the difference equation for $d(k)$.

Theorem 4 Let $d(k)$ be the degree of the mapping Φ_n^k

$$\Phi_n^k : z \rightarrow z' \quad z'_1 : z'_2 : z'_3 = \phi_1^{(k)}(z) : \phi_2^{(k)}(z) : \phi_3^{(k)}(z), \quad (49)$$

the k^{th} iteration of the mapping Φ_n (2) of characteristic $\text{char}(\Phi_n) = \{n, i_1, \dots, i_\sigma\}$, O_{α_j} and $O_{\beta_j}^{(-1)}$, $j = 1, \dots, \sigma_1$ be σ_1 pairs of equivalent indeterminacy points of the subsets $O^{(int)}$ and $O^{(-1)(int)}$ defined by relations (46), (47) and (48) (see Section 3, Definitions 13 and 14). Let also $\gamma_{\alpha_j}(k)$ be common multiplicities of the curves $\{\phi_i^{(k)}(z) = 0, i = (1, 2, 3)\}$ and the general curve of the linear system $\phi_\mu^{(k)}(z) = \{\sum_{i=1}^{i=3} c_i \phi_i^{(k)}(z) = 0, \forall c_i \in C\}$ of degree $\mu = d(k)$ at indeterminacy points O_{α_j} of the direct mapping (2) (we assume that all F-points are already ordinary after the birational equivalence transformation – see Theorem 3). Then the dynamics of the mapping Φ_n (2) (see Definition 5, Remarks 2 and 3) is completely determined by the following set of difference equations:

$$d(k) = nd(k-1) - \sum_{l=1}^{\sigma_1} i_{\beta_l}^{(-1)} \gamma_{\alpha_l}(k-m_l-1), \quad (50)$$

$$\gamma_{\alpha_j}(k) = i_{\alpha_j} d(k-1) - \sum_{l=1}^{\sigma_1} i_{\alpha_j \beta_l} \gamma_{\alpha_l}(k-m_l-1), \quad j = 1, \dots, \sigma_1, \quad (51)$$

$$\gamma_\alpha(k) = i_\alpha d(k-1) - \sum_{l=1}^{\sigma_1} i_{\alpha \beta_l} \gamma_{\alpha_l}(k-m_l-1), \quad \alpha \neq \alpha_j, \quad (52)$$

moreover,

$$d(0) = 1, \quad d(1) = n, \quad \gamma_\alpha(1) = i_\alpha, \quad \gamma_\alpha(k) = 0 \quad \text{for } k \leq 0. \quad (53)$$

The secular equation corresponding to the set of difference equations (50)-(51) is

$$\det(\Lambda) = \lambda^m + \sum_{i=0}^{m-1} a_i \lambda^i = 0, \quad (54)$$

where integer m is

$$m = m_1 + m_2 + \dots + m_{\sigma_1} + \sigma_1 + 1, \quad (55)$$

integers a_i are coefficients of expansion in power series of $\det(\Lambda)$ in λ and the matrix Λ is

$$\Lambda = \begin{pmatrix} \lambda - n, & i_{\beta_1}^{(-1)}, & \dots, & i_{\beta_{\sigma_1}}^{(-1)} \\ -i_{\alpha_1}, & (\lambda^{m_1+1} + i_{\alpha_1 \beta_1}), & \dots, & i_{\alpha_1 \beta_{\sigma_1}} \\ \vdots & \vdots & \ddots & \vdots \\ -i_{\alpha_{\sigma_1}}, & i_{\alpha_{\sigma_1} \beta_1}, & \dots, & (\lambda^{m_{\sigma_1}+1} + i_{\alpha_{\sigma_1} \beta_{\sigma_1}}) \end{pmatrix}. \quad (56)$$

The linear difference equation for $d(k)$ corresponding to the secular equation (54) has the form

$$d(k+m) + \sum_{i=0}^{m-1} a_i d(k+i) = 0, \quad (57)$$

where a_i are the same integers as in equation (54).

According to a general theory of linear difference equations with constant coefficients (Gel'fond, 1971) (Chapter V), the solution of equation (57) has the form

$$d(k) = \sum_{i=1}^l \lambda_i^k \left(\sum_{j=0}^{s_i-1} c_{ij} k^j \right), \quad (58)$$

where $\lambda_1, \dots, \lambda_l$ are multiple roots of equation (54) with multiplicities s_1, \dots, s_l , $s_1 + s_2 + \dots + s_l = m$, and c_{ij} are arbitrary constants to be determined from m initial values $d(1) = n, d(2), \dots, d(m)$ obtained with the help of equations (50)-(53).

It is obvious that the condition

$$|\lambda_i| = 1 \quad \forall i \in (1, 2, \dots, l) \quad (59)$$

is sufficient for the polynomial boundedness of the growth of $d(k)$ with k .

Remark 6 If some integers m_i are equal and, moreover, are fulfilled corresponding conditions for the coefficients at the terms $\gamma_{\alpha_i}(k)$ in equations (50)-(51), we can decrease the order of the system of difference equations (50)-(51) and, as result, the order of the matrix Λ .

◇

Proof. Let us prove the theorem by an induction method. Let us consider the map Φ_n^k (49)-the k^{th} iteration of the mapping Φ_n (2) as an iteration of the map Φ_n^{k-1} and let us consider the transformation of a general curve of a linear system of the curves $\phi_\mu^{(k-1)}(z) = \sum_{i=1}^{i=3} c_i \phi_i^{(k-1)}(z) = 0$ of degree $\mu = d(k-1)$ by the action of the mapping Φ_n (2). Let $\gamma_\beta^{(-1)}(k-1)$ be common multiplicities of the curves $\{\phi_i^{(k-1)}(z) = 0, \forall i \in (1, 2, 3)\}$ and of the general curve of the linear system $\{\phi_\mu^{(k-1)}(z) = 0\}$ at the indeterminacy points $O_\beta^{(-1)}$ of the inverse map $\Phi_n^{(-1)}$ (4). Then, according to Subsection 2.4, Remark 4, (37) and (38), we have

$$\phi_\mu^{(k-1)}(\phi(z)) = \phi_{\mu'}^{(k)}(z) \prod_{\beta=1}^{\sigma} j_\beta^{\gamma_\beta^{(-1)}(k-1)}(z), \quad (60)$$

$$\mu' = \mu n - \sum_{\beta=1}^{\sigma} i_\beta^{(-1)} \gamma_\beta^{(-1)}(k-1), \quad (61)$$

$$\gamma_\alpha(k) = \mu i_\alpha - \sum_{\beta=1}^{\sigma} i_{\alpha\beta} \gamma_\beta^{(-1)}(k-1), \quad (62)$$

where $\phi_{\mu'}^{(k)}(z) = \sum_{i=1}^{i=3} c_i \phi_i^{(k)}(z)$ is a general curve of a linear system of curves of degree $\mu' = d(k)$ associated with the map Φ_n^k but $\gamma_\alpha^{(k)}$ are its multiplicities at the points O_α . Since the linear system of the curves $\{\phi_\mu^{(k-1)}(z) = 0\}$ is completely defined by its basis set, the difference of values $\gamma_\beta^{(-1)}(k-1)$ from zero means that the set $O^{(k-1)} \cap O^{(-1)} \neq \emptyset$ where $O^{(k-1)}$ is the set of indeterminacy points the mapping Φ_n^{k-1} . It is obvious that the set $O^{(k-1)}$ is equal to $(O^{(1)} \equiv O)$

$$O^{(k-1)} = \bigcup_{l=0}^{l=k-2} \Phi_n^{-l}(O(l)), \quad (63)$$

where the set $O(l)$ is defined by equation (43) (see Section 3). Let us decompose the set $O^{(k-1)}$ into two subsets

$$O^{(k-1)} = O^{(k-1)(\text{int})} \bigcup O^{(k-1)(\text{rest})} \quad (64)$$

related with the subsets $O^{(int)}$ and $O^{(rest)}$ in equation (39). Then the subset $O^{(k-1)(int)}$ is equal to

$$O^{(k-1)(int)} = \bigcup_{j=1}^{j=\sigma_1} \bigcup_{l=0}^{l=\min(m_j, k-2)} \Phi_n^{-l}(O_{\alpha_j}), \quad (65)$$

according to Section 3, Definitions 13, 14 and (46), (47) and (48).

Since, according to Section 3, the intersection of the subsets $O^{(k-1)(rest)}$ and $O^{(-1)}$ is empty, then $O^{(k-1)} \cap O^{(-1)} = O^{(k-1)(int)} \cap O^{(-1)}$ and

$$O^{(k-1)} \cap O^{(-1)} = \left\{ \begin{array}{ll} \bigcup_{j=1}^{j=\sigma_1} \Phi_n^{-m_j}(O_{\alpha_j}) & = \bigcup_{j=1}^{j=\sigma_1} O_{\beta_j}^{(-1)}, & \forall (k-2) \geq m_j, \\ \emptyset & & \forall (k-2) < m_j. \end{array} \right\} \quad (66)$$

Then, according to (66), we have for $\gamma_{\beta}^{(-1)}(k-1) = \text{mult}(\phi_{\mu}^{(k-1)}(z)|_{z=O_{\beta}^{(-1)}})$

$$\gamma_{\beta}^{(-1)}(k-1) = \left\{ \begin{array}{ll} 0, & \forall \beta \neq \beta_j, (\beta, \beta_j) \in (1, \dots, \sigma) \\ \gamma_{\alpha_j}(k-1-m_j), & \beta \equiv \beta_j, j \in (1, \dots, \sigma_1) \end{array} \right\}, \quad (67)$$

where

$$\gamma_{\alpha}(k) = 0, \quad k \leq 0, \quad \gamma_{\alpha}(1) \stackrel{\text{def}}{=} i_{\alpha}. \quad (68)$$

At last, substituting (67) into equations (60)-(62) and taking into account (68) and $d(0) = 1, d(1) = n$, we have equations (50)-(53). Equations (50)-(53) hold at $k = 1$ and, therefore, at $\forall k > 1$. Let us prove the validity of equations (54)-(59). There are two ways of obtaining them. Let us transform the system of $\sigma_1 + 1$ equations (50) and (51) at $\alpha_j = (\alpha_1, \alpha_2, \dots, \alpha_{\sigma_1})$ by changing $k \rightarrow k + m_1 + 1$ to the following form ($j = 1, \dots, \sigma_1$):

$$\sum_{j=1}^{\sigma_1} i_{\beta_j}^{(-1)} \gamma_{\alpha_j}(k + m_1 - m_j) = nd(k + m_1) - d(k + m_1 + 1) \quad (69)$$

$$\gamma_{\alpha_j}(k + m_1 + 1) + \sum_{l=1}^{\sigma_1} i_{\alpha_j \beta_l} \gamma_{\alpha_l}(k + m_1 - m_l) = d(k + m_1) i_{\alpha_j}. \quad (70)$$

The first way is to obtain a linear homogeneous difference equation (57) for $d(k)$ excepting from the homogeneous system of $\sigma_1 + 1$ difference equations (69), (70) for unknown $d(k), \gamma_{\alpha_j}(k), j \in (1, 2, \dots, \sigma_1)$ step by step $\gamma_{\alpha_1}(k), \gamma_{\alpha_2}(k), \dots$ and so on, until we do not obtain equation (57) for $d(k)$.

However, we can present a more direct method of obtaining equation (57) for the function $d(k)$ through finding the characteristic or secular equation immediately from the system of equations (69)-(70) performing substitution in them accordingly to

$$d(k) = b_0 \lambda^k, \quad \gamma_{\alpha_j}(k) = b_j \lambda^k, \quad j \in (1, 2, \dots, \sigma_1) \quad (71)$$

where $b_0, b_j, j \in (1, 2, \dots, \sigma_1)$ are unknown constants.

After the substitution (71) the system of equations (69)-(70) has the following matrix form:

$$\Lambda D B = 0, \quad D = \text{diag}(\lambda^{m_1}, \lambda^{m_1-m_2}, \dots, \lambda^{m_1-m_{\sigma_1}}), \quad B = (b_0, b_1, \dots, b_{\sigma_1}), \quad (72)$$

where the matrix Λ is defined by equation (56).

The compatibility condition of the homogeneous system (72) with respect to unknown parameters $b_0, b_j, j \in (1, 2, \dots, \sigma_1)$ is a secular equation (54) where integers a_i , the same as in eq. (57), are the coefficients of expansion in power series of $\det(\Lambda)$ in λ but integer m is defined by (55). Since the difference equation for $d(k)$ (57) and the secular equation (54) are in one-to-one correspondence by the substitution $d(k) = \lambda^k$, we can uniquely reconstruct (57) from (54).

At the end, according to the general theory of linear difference equations with constant coefficients (see (Gel'fond, 1971), chapter V), the general solution of equation (57) is completely defined by the spectrum of eigenvalues of the characteristic secular equation (54) and this solution has the form (58). Remark that the method of obtaining a general solution of the system of difference equations (50), (51) offered above is fully analogous to usual practice of solving a system of linear differential equations of an order more one with constant coefficients (see, for example, (Arnold, 1984), the chapter 3, §25). \triangleleft

5 Conclusion

We believe in that all mappings with the Arnold complexity defined by equation (58) and the spectrum (59) of the secular equation (54) of the order m (55) with the matrix Λ (56) are algebraically integrable ones and are intended to prove this theorem.

Theorem 4 gives us to possibility to generate, for example, all integrable families of maps of degree $n = 2$ in the parameter space of dimension $2\sigma + 8 - 2\sigma_1 = 14 - 2\sigma_1$ being stratified on algebraic subvarieties in this space. We can present here some different interesting sets for $n = 2, \sigma_1 = 3, \alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_3$:

$$m_1 = 0, m_2 \neq m_3 \neq 0, \det(\Lambda) = (\lambda - 1)^2(\lambda^{m_2+m_3+2} - 1); \quad (73)$$

$$m_1 = 1, m_2 = 2, m_3 = 3, \det(\Lambda) = (\lambda - 1)^3(\lambda + 1) \frac{(\lambda^9 + 1)}{(\lambda^3 + 1)}; \quad (74)$$

$$m_1 = 1, m_2 = 2, m_3 = 4, \det(\Lambda) = (\lambda - 1)^3 \left[\frac{(\lambda^{15} + 1)(\lambda + 1)}{(\lambda^5 + 1)(\lambda^3 + 1)} \right]; \quad (75)$$

$$m_1 = 1, m_2 = 2, m_3 = 5, \det(\Lambda) = (\lambda - 1)^3(\lambda + 1)(\lambda^5 - 1)(\lambda^3 - 1); \quad (76)$$

$$m_1 = 0, m_2 = m_3 = m \geq 1, \det(\Lambda) = (\lambda - 1)^2(\lambda^{m+1} + 1); \quad (77)$$

$$m_1 = 1, m_2 = m_3 = 2, \det(\Lambda) = (\lambda - 1)^2 \left[\frac{\lambda^6 + 1}{\lambda^2 + 1} \right]; \quad (78)$$

$$m_1 = m_2 = m_3 = 1, \det(\Lambda) = (\lambda - 1) \frac{\lambda^3 + 1}{\lambda + 1}; \quad (79)$$

$$m_1 = m_2 = m_3 = 2, \det(\Lambda) = (\lambda - 1)^3(\lambda + 1). \quad (80)$$

It isn't difficult to obtain polynomially abounded dependence $d(k)$ for sets (73)–(80)

One presents itself interesting also to give classification of all integrable cubic maps.

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7 Appendix A. Examples of maps and FEs.

1. FE of the paper.

$$y(w+2) = \frac{y(w+1)(\lambda y(w+1) + dy(w))}{y(w)}, \quad (81)$$

Supposing $y(w+2) = y'_2, y(w+1) = y'_1 = y_2, y(w) = y_1$ we have the map $y \mapsto y' : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ and then changing $y \mapsto z : y_i = z_i/z_3$ we obtain the map $z \mapsto z' : \mathbf{CP}^2 \rightarrow \mathbf{CP}^2$:

$$\Phi_2 : z'_1 : z'_2 : z'_3 = z_2 z_1 : z_2(\lambda z_2 + dz_1) : z_1 z_3, \quad (82)$$

$$\Phi_2^{(-1)} : z_1 : z_2 : z_3 = \lambda z'_1^2 : z'_1(z'_2 - dz'_1) : z'_3(z'_2 - dz'_1), \quad (83)$$

$$Jac(\Phi_2) = 2\lambda z_1 z_2^2, \quad Jac(\Phi_2^{(-1)}) = 2\lambda z'_1^2(z'_2 - dz'_1), \quad (84)$$

$$\begin{aligned} O_1 &= (1, 0, 0), \quad O_2^* = O_3^* = (0, 0, 1), \\ J_1 &: (z_1 = 0), \quad J_2 = J_3 : (z_2 = 0), \end{aligned} \quad (85)$$

$$\begin{aligned} O_1^{(-1)} &= (0, 1, 0), \quad O_2^{(-1)*} = O_3^{(-1)*} = (0, 0, 1), \\ J_1^{(-1)} &: (z'_2 - dz'_1 = 0), \quad J_2^{(-1)} = J_3^{(-1)} : (z'_1 = 0). \end{aligned} \quad (86)$$

2. the FE (90) from (Rerikh, 1992) $F(w+1) = \frac{3F(w)-F(w-1)+F(w)F(w-1)}{1+F(w)}$.

Omitting changes (see previous example) we have

$$\Phi_2 : z'_1 : z'_2 : z'_3 = z_2(z_2 + z_3) : 3z_2z_3 - z_1z_3 + z_1z_2 : z_3(z_2 + z_3), \quad (87)$$

$$\Phi_2^{(-1)} : z_1 : z_2 : z_3 = 3z'_1z'_3 - z'_2z'_3 - z'_1z'_2 : z'_1(z'_3 - z'_1) : z'_3(z'_3 - z'_1), \quad (88)$$

$$\text{Jac}(\Phi_2) = 2(z_3 - z_2)(z_2 + z_3)^2, \quad \text{Jac}(\Phi_2^{(-1)}) = 2(z'_1 + z'_3)(z'_3 - z'_1)^2, \quad (89)$$

$$O_1 = (-3/2, -1, 1), \quad O_2^* = O_3^* = (1, 0, 0), \quad (90)$$

$$J_1 : (z_3 - z_2 = 0), \quad J_2 = J_3 : (z_3 + z_2 = 0), \quad (90)$$

$$O_1^{(-1)} = (1, 3/2, 1), \quad O_2^{(-1)*} = O_3^{(-1)*} = (0, 1, 0), \quad (90)$$

$$J_1^{(-1)} : (z'_1 + z'_3 = 0), \quad J_2^{(-1)} = J_3^{(-1)} : (z'_3 - z'_1 = 0) \quad (91)$$

3. Mapping of the paper.

$$\begin{aligned} \Phi_2 : z'_1 : z'_2 : z'_3 &= [z_1z_3 - p_2z_1^2 + \frac{(q_2 + q_3)}{2}z_1z_2 + \frac{(q_2 - q_3)^2}{12p_2}z_2^2] : \\ &\quad -z_2[z_3 + 2p_2z_1 + \frac{q_2 + q_3}{2}z_2] : z_3^2 + \frac{3}{2}(q_2 + q_3)z_2z_3 \\ &\quad -p_2^2z_1^2 + p_2(q_2 + q_3)z_1z_2 + \frac{1}{12}(5q_2^2 + 14q_2q_3 + 5q_3^2)z_2^2, \end{aligned} \quad (92)$$

$$\Phi_2^{-1} = \Lambda \circ \Phi_2 \circ \Lambda, \quad \Lambda = \text{diag}(-1, 1, 1). \quad (93)$$

The mapping (92) follows from generic quadratic map (18) if we suppose $B_1 = B\Lambda$, $\Lambda = \text{diag}(-1, 1, 1)$ and

$$B = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}, \quad (94)$$

where

$$p_1 = -2p_2, \quad p_3 = p_2, \quad q_1 = (q_2 + q_3)/2, \quad r_1 = r_2 = r_3 = 1. \quad (95)$$

In accordance with formulae (18), (19) and (24) we have three principal lines $J_i, J_i^{(-1)}$ and three F-points $O_i, O_i^{(-1)}$, $O_i = (j_j = 0) \cap (j_k = 0)$, $O_i^{(-1)} = (j_j^{(-1)} = 0) \cap (j_k^{(-1)} = 0)$, $i \neq j \neq k$, $i, j, k \in \{1, 2, 3\}$:

$$\begin{aligned} J_i &: (j_i = -p_iz_1 + q_iz_2 + r_iz_3 = 0), \\ J_i^{(-1)} &: (j_i^{(-1)} = p_iz_1 + q_iz_2 + r_iz_3 = 0), \end{aligned} \quad (96)$$

$$O_i = \{q_j - q_k, p_j - p_k, p_kq_j - p_jq_k\}, \quad O_i^{(-1)} = \Lambda O_i, \quad (97)$$

$$\begin{aligned} O_1 &= \left(\frac{1}{p_2}, 0, 1 \right), \quad O_2 = \left(-\frac{q_3 - q_2}{p_2(5q_3 + q_2)}, -\frac{6}{5q_3 + q_2}, 1 \right), \\ O_3 &= \left(\frac{q_3 - q_2}{p_2(5q_2 + q_3)}, -\frac{6}{5q_2 + q_3}, 1 \right). \end{aligned} \quad (98)$$

4. FE $F(w+1) = \frac{4+2F(w)F(w-1)+F(w-1)-14F^2(w)-4F(w-1)F^2(w)}{1-2F(w)-2F(w-1)-4F^2(w)}$ of paper (Rerikh, 1995b)

Let us consider the cubic birational mapping $\Phi_3 : \mathbf{CP}^2 \mapsto \mathbf{CP}^2$ associated with the above functional equation from (Rerikh, 1995b) (see eq. 23 on p. 67 and eq. 30 on p. 68)

$$\begin{aligned} \Phi_3 &: z'_1 : z'_2 : z'_3 = z_2(z_3^2 - 2z_1z_3 - 2z_2z_3 - 4z_2^2) : (4z_3^3 + z_1z_3^2 + \\ &\quad 2z_1z_2z_3 - 14z_2^2z_3 - 4z_1z_2^2) : z_3(z_3^2 - 2z_1z_3 - 2z_2z_3 - 4z_2^2), \end{aligned} \quad (99)$$

$$\begin{aligned} \Phi_3^{(-1)} &: z_1 : z_2 : z_3 = -(4z_3'^3 - z_2'z_3'^2 + 2z_1'z_2'z_3' - 14z_1'^2z_3' + 4z_2'z_1'^2) : \\ &\quad z_1'(z_3^2 + 2z_1'z_3' + 2z_2'z_3' - 4z_1'^2) : \\ &\quad z_3'(z_3^2 + 2z_1'z_3' + 2z_2'z_3' - 4z_1'^2), \end{aligned} \quad (100)$$

where $y_1 = F(w - 1)$, $y'_1 = y_2 = F(w)$, $y'_2 = F(w + 1)$ and $y_i = \frac{z_i}{z_3}$.

These maps ($\text{char}=\{3;2,1,1,1,1\}$) have the following indeterminacy points and principal curves:

$$\begin{aligned} O_1 &= (1, 0, 0), \quad O_2 = (-1/2, 1/2, 1), \quad O_3 = (-5/2, -3/2, 1), \\ O_4 &= (1/2, -1/2, 1), \quad O_5 = (-11/2, 3/2, 1), \end{aligned} \quad (101)$$

$$\begin{aligned} O_1^{(-1)} &= (0, 1, 0), \quad O_2^{(-1)} = (3/2, 5/2, 1), \quad O_3^{(-1)} = (-1/2, 1/2, 1), \\ O_4^{(-1)} &= (-3/2, 11/2, 1), \quad O_5^{(-1)} = (1/2, -1/2, 1), \end{aligned} \quad (102)$$

$$\begin{aligned} J_1 &: (z_3^2 - 2z_1z_3 - 2z_2z_3 - 4z_2^2 = 0), \quad J_2 : (2z_2 - 3z_3 = 0), \\ J_3 &: (z_3 + 2z_2 = 0), \quad J_4 : (2z_2 + 3z_3 = 0), \\ J_5 &: (z_3 - 2z_2 = 0), \end{aligned} \quad (103)$$

$$\begin{aligned} J_1^{(-1)} &: (z_3'^2 + 2z_1'z_3' + 2z_2'z_3' - 4z_1'^2 = 0), \quad J_2^{(-1)} : (2z_1 - z_3 = 0), \\ J_3^{(-1)} &: (2z_1 + 3z_3 = 0), \quad J_4^{(-1)} : (2z_1 + z_3 = 0), \\ J_5^{(-1)} &: (2z_1 - 3z_3 = 0). \end{aligned} \quad (104)$$

It is not difficult to obtain $i_{\alpha,\beta}$:

$$i_{\alpha,\beta} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (105)$$

8 Appendix B. Examples to Sections 3 and 4

Below we return again to Examples 1-4 (see Appendix A) for illustration of Sections 3 and 4. Firstly, we give the set $O^{(int)}$ derived with the help of the decomposition procedure, Section 3 (39), (40), and then give the equations of the dynamics (50), (51) of maps 1-4, and finish with a set of difference equation for the Arnold complexity and its solution.

- This map (82) has two infinitely near points O_2^* and O_3^* . (O_2^* is merged with O_3^* in the direction J_1 . The point O_1 belongs $O^{(inf)}$: $\Phi_2^{(-k)}(O_1) = (\lambda^k, -d\frac{\lambda^k-1}{\lambda-1}, 0)$, but, say, the point $O_3^* \in O^{(int)}$: $O_3^* = O_3'^*$). We can ascertain that the point O_2^* does not belong to $O^{(int)}$: if we consider a map $\Phi_{2,\epsilon}$ being a small deformation of initial map (82) and having three different indeterminacy points ($O_1 = (1, 0, 0)$, $O_2 = (0, \epsilon/\lambda, 1)$, $O_3 = (0, 0, 1)$)

$$\Phi_{2,\epsilon} : z'_1 : z'_2 : z'_3 = z_2z_1 : z_2(\lambda z_2 - \epsilon z_3 + dz_1) : z_1z_3, \quad (106)$$

Since it is difficult to point out a general method of constructing a map coinciding with a given map at a small parameter $\epsilon = 0$ and having only ordinary indeterminacy points, we change the given map

by a birationally equivalent map with ordinary indeterminacy points. The map $\Phi_2 : z \mapsto z'$, $z, z' \in \mathbb{CP}^2$ (82) is birationally equivalent to the map $\Phi_3 : u \mapsto u'$, $u' \in \mathbb{CP}^2$, $\Phi_3 = \Psi_2^{(-1)} \circ \Phi_2 \circ \Psi_2$

$$\begin{aligned} \Phi_3 : u'_1 : u'_2 : u'_3 &= [(\lambda - d)u_1 + du_2] [-u_3((\lambda - d)u_1 + du_2) + \\ &\quad (u_2 - u_1)(u_3 + u_2)] : [(1 + d)u_2 - (1 + d - \lambda)u_1] [-u_3((\lambda - d)u_1 + du_2) + \\ &\quad (u_2 - u_1)(u_3 + u_2)] : \\ &\quad u_3[(\lambda - d)u_1 + du_2][(1 + d)u_2 - (1 + d - \lambda)u_1], \end{aligned} \quad (107)$$

$$\begin{aligned} \Phi_3^{(-1)} : u_1 : u_2 : u_3 &= [u'_1(d + 1) - du'_2][2u'_1u'_3 + u'_1u'_2 - u'_2u'_3] : \\ &\quad [u'_1(d + 1 - \lambda) - (d - \lambda)u'_2][2u'_1u'_3 + u'_1u'_2 - u'_2u'_3] : \\ &\quad u'_3(u'_2 - u'_1)[u'_1(d + 1 - \lambda) - (d - \lambda)u'_2], \end{aligned} \quad (108)$$

where the map $\Psi_2 : u \mapsto z$ is chosen so that two indeterminacy points of the map $\Psi_2^{(-1)}$ may coincide with the points O_1, O_2^* from (85), but a direction of the second principal curve (line) for $\Psi_2^{(-1)}$ through the point O_2^* does not coincide with J_1 from (85):

$$\Psi_2 : u \mapsto z \ z_1 : z_2 : z_3 = (u_2u_3 - u_1u_3) : u_1u_3 : (-u_1u_3 + u_1u_2). \quad (109)$$

The maps $\Phi_3, \Phi_3^{(-1)}$ have $\text{char} = \{2, 1, 1, 1, 1\}$ and the following indeterminacy points:

$$\begin{aligned} O_1 &= (0, 0, 1), \quad O_2 = (1, 1, 0), \quad O_3 = (1, 0, 0), \\ O_4 &= \left(-\frac{d}{d - \lambda}, -1, 1\right), \quad O_5 = \left(-\frac{2(1 + d)}{1 - \lambda + d}, -2, 1\right), \\ O_1^{(-1)} &= (0, 0, 1), \quad O_2^{(-1)} = (1, 0, 0), \quad O_3^{(-1)} = (0, 1, 0), \\ O_4^{(-1)} &= \left(-\frac{d - \lambda - 1}{d - \lambda + 1}, -\frac{d - \lambda - 1}{d - \lambda}, 1\right), \quad O_5^{(-1)} = (-1, -1, 1). \end{aligned}$$

Making the decomposition we obtain

$$\begin{aligned} O_1, O_3 &\in \mathcal{O}^{(int)}, \quad \Phi_3^{(-1)}O_2 = O_2, \quad O_2 \in \mathcal{O}^{(cycle)}, \quad O_4, O_5 \in \mathcal{O}^{(inf)}, \\ O_1^{(-1)}, O_2^{(-1)} &\in \mathcal{O}^{(-1)(int)}, \quad O_3^{(-1)}, O_4^{(-1)} \in \mathcal{O}^{(-1)(inf)}, \quad O_5^{(-1)} \in \mathcal{O}^{(-1)(cycle)}. \end{aligned}$$

Following Theorem 4 we have ($i_{\alpha\beta}$ is the same one as in example 4 (105))

$$d(k) = 3d(k - 1) - 2\gamma_1(k - 1) - \gamma_3(k - 1), \quad (110)$$

$$\gamma_1(k) = 2d(k - 1) - \gamma_1(k - 1) - \gamma_3(k - 1), \quad (111)$$

$$\gamma_3(k) = d(k - 1) - \gamma_1(k - 1). \quad (112)$$

Following Theorem 4 we obtain

$$d(k + 2) - 2d(k + 1) + d(k) = 0, \quad d(k) = 2k + 1. \quad (113)$$

2. The map Φ_2 (87) is birationally equivalent to the map $\Phi_2^* = \Psi_2^{(-1)} \circ \Phi_2 \circ \Psi_2$ with ordinary indeterminacy points (conditions for the choice of Ψ_2 : $O_1^{(-1)}(\Psi_2^{(-1)}) = O_{2,3}(\Phi_2)$, $O_2^{(-1)}(\Psi_2^{(-1)}) = O_{2,3}^{(-1)}(\Phi_2^{(-1)})$, $J_1^{(-1)}, J_1$ for Φ_2 from (90), (91) must not coincide with J_i for Ψ_2)

$$\begin{aligned} \Phi_2^* : u'_1 : u'_2 : u'_3 &= u_1(3u_2 - u_3 + u_1) : u_3(3u_2 - u_3 + u_1) : \\ &\quad u_2(u_1 + u_3), \end{aligned} \quad (114)$$

$$\begin{aligned} \Phi_2^{*(-1)} : u_1 : u_2 : u_3 &= u'_1(u'_2 + u'_1 - 3u'_3) : u'_3(u'_1 - u'_2) : \\ &\quad u'_2(u'_2 + u'_1 - 3u'_3), \end{aligned} \quad (115)$$

but the map Ψ_2 is $\Psi_2 : z_1 : z_2 : z_3 = u_1 u_3 : u_1 u_2 : u_2 u_3$.

We have the indeterminacy points for Φ_2^* : $O_1 = (-1, \frac{2}{3}, 1)$, $O_2 = (1, 0, 1)$, $O_3 = (0, 1, 0)$, $O_1^{(-1)} = (\frac{3}{2}, \frac{3}{2}, 1)$, $O_2^{(-1)} = (-1, 1, 0)$, $O_3^{(-1)} = (0, 0, 1)$.

The decomposition of the sets $O, O^{(-1)}$ gives: $O_1, O_2 \in O^{(inf)}$, $O_1^{(-1)}, O_2^{(-1)} \in O^{(-1)(inf)}$, $\Phi_2^{*(-1)}(O_3) = O_3^{(-1)}$, $O_3 \in O^{(int)}$, $O_3^{(-1)} \in O^{(-1)(int)}$.

We have for the Arnold complexity $d(k) = 2d(k-1) - \gamma_3(k-2)$, $\gamma_3(k) = d(k-1)$, $d(k+3) - 2d(k+2) + d(k) = 0$, $d(k) = -1 + (\lambda^{k+3} + (-1)^k \lambda^{-(k+3)})/\sqrt{5}$, where $\lambda = \frac{\sqrt{5}+1}{2}$.

3. The decomposition of the sets $O, O^{(-1)}$ of indeterminacy points $O_\alpha, O_\beta^{(-1)}$ (97), (98) of the mappings Φ_2 (92) and $\Phi_2^{(-1)}$ (93) gives

$$\begin{aligned}\Phi_2^{-1}(O_1) &= \{\infty, 0\}, & \Phi_2^{-2}(O_1) &= O_1^{(-1)}, \\ \Phi_2^{-1}(O_2) &= \left\{ \frac{q_3 - q_2}{3p_2(q_3 + q_2)}, -\frac{2}{q_3 + q_2} \right\}, & \Phi_2^{-2}(O_2) &= O_3^{(-1)}, \\ \Phi_2^{-1}(O_3) &= \left\{ -\frac{q_3 - q_2}{3p_2(q_3 + q_2)}, -\frac{2}{q_3 + q_2} \right\}, & \Phi_2^{-2}(O_3) &= O_2^{(-1)},\end{aligned}\quad (116)$$

and, consequently, $O \equiv O^{(int)}$, $O^{(-1)} \equiv O^{(-1)(int)}$, $m_j = 2 \forall j \in (1, 2, 3)$, $\alpha_j = (1, 2, 3)$, $\beta_j = (1, 3, 2)$.

Due to Theorem (4) and Remark (6) we have from (49)- (54)

$$d(k) = 2d(k-1) - S(k-3), \quad (117)$$

$$S(k) = 3d(k-1) - 2S(k-3), \quad (118)$$

where $S(k) = \sum_{\alpha=1}^3 \gamma_\alpha(k)$. In correspondence with Theorem 4 we obtain the difference equation for $d(k)$

$$d(k+4) - 2d(k+3) + 2d(k+1) - d(k) = 0 \quad (119)$$

and its general solution

$$d(k) = \frac{3}{4}k^2 - \frac{1}{8}(-1)^k + \frac{9}{8}. \quad (120)$$

4. The decomposition of the sets $O, O^{(-1)}$ of indeterminacy points $O_\alpha, O_\beta^{(-1)}$ (101), (102) of the mappings Φ_3 (99) and $\Phi_3^{(-1)}$ (100) gives:

$$\begin{aligned}O_2 &= O_3^{(-1)}, \quad \alpha_1 = 2, \beta_1 = 3, \quad m_1 = 0, \\ O_4 &= O_5^{(-1)}, \quad \alpha_2 = 4, \beta_2 = 5, \quad m_2 = 0, \\ \Phi_3^{(-1)}(O_1) &= O_1^{(-1)}, \quad \alpha_3 = 1, \beta_3 = 1, \quad m_3 = 1, \\ O^{(int)} &= \{O_2, O_4, O_1\}, \quad O^{(-1)(int)} = \{O_3^{(-1)}, O_5^{(-1)}, O_1^{(-1)}\}, \\ O^{(inf)} &= \{O_3, O_5\}, \quad O^{(-1)(inf)} = \{O_2^{(-1)}, O_4^{(-1)}\}.\end{aligned}\quad (121)$$

Following Theorem 4 (see (54) and (56)) we have $\text{Det}(\Lambda) = (\lambda - 1)^2(\lambda^3 - \lambda^2 - \lambda - 1)$ and $d(k) = c_{00} + c_{01}k + \sum_{i=1}^{i=3} c_{1i} \lambda_i^k$, where the coefficients c_{0i} and c_{1i} are defined in terms of λ_i —the roots of the cubic equation $\lambda^3 - \lambda^2 - \lambda - 1 = 0$.

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